UCD Enrichment Programme in Mathematics
Solutions Selection Test 27 February 2016

1. I have two egg timers. The first can time an interval of exactly 7 minutes. The second can time an interval of exactly 9 minutes. Explain how I can use them to boil an egg for exactly 3 minutes?

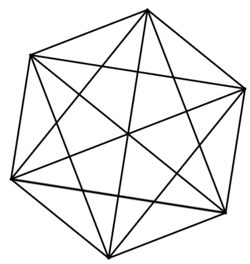
   Solution. We have $3 \cdot 7 - 2 \cdot 9 = 3$.

   Start both timers at the same time. When either expires, reset it.

   Place the egg in the boiling water when the 9-minute timer reaches the end of its second cycle.

   Remove the egg from the water when the 7-minute timer reaches the end of its third cycle.

2. Consider the hexagon shown below. Alternately, two players play the following game: in one move one player selects



   one edge and colours it red and then the other player selects one remaining edge and colours it green. The winner is the player to complete first a triangle in their colour.

   Show that the player who goes first has a strategy that will always guarantee a win in four moves.

   Solution. Denote the vertices by A,B,C,D,E,F. First player picks a vertex A and colours A to B red in the first move, A to C red in the second move. Other player must colour B to C green in their second move. First player colours A to D red in their third move. Fourth move is either B to D or C to D.

   If the second player colours AC green in his first move, then the first player should colour BF and BE in his next moves.

3. (a) Show that the greatest common divisor of $(n+1)! + 1$ and $n! + 1$ is 1, for all integers $n \geq 1$.

   (b) For any $n > 1$, find integers x, y such that

   $$((n+1)! + 1)x + (n! + 1)y = 1.$$

   [Recall that $n! = 1 \times 2 \times \cdots \times n$ for any $n > 1$.]

   Solution. (Of course, (b) implies (a), but we prove (a) first.)

   (a) Suppose that $d \geq 1$ is a common divisor of $a = (n+1)! + 1$ and $b = n! + 1$.
   Then $d|(n+1)b - a = n$ (using $(n+1)n! = (n+1)!$).
   Hence $d|b - n(n-1)! = b - n! = 1$. So $d = 1$.

   (b) From the solution to part (a), we have

   $$(n+1)b - a = n$$
   $$b - (n-1)!n = 1.$$

   Thus

   $$\begin{aligned} 1 &= b - (n-1)!n \\ 1 &= b - (n-1)!((n+1)b - a) \\ 1 &= (n-1)!a + (1 - (n-1)!(n+1))b \end{aligned}$$

   Thus $x = (n-1)!$, $y = 1 - (n-1)!(n+1)$ are the requested integers.
   (Alternatively, apply Euclid's algorithm to a and b.)

4. Prove that for any positive real numbers a, b and c we have

$$\frac{2a+b}{b+2c}+\frac{2b+c}{c+2a}+\frac{2c+a}{a+2b}\geq 3.$$

Solution. Let

$$\begin{aligned} b+2c &= x \qquad (1)\\ c+2a &= y \qquad (2)\\ a+2b &= z \qquad (3)\end{aligned}$$

Adding the above equalities we find

$$2a+2b+2c=\frac{2(x+y+z)}{3} \qquad (4)$$

Now, from (1) and (4) we find

$$2a+b=\frac{2y+2z-x}{3}$$

and similarly,

$$2b+c=\frac{2x+2z-y}{3} \qquad \text{and} \qquad 2c+a=\frac{2x+2y-z}{3}.$$

Thus, in the new variables x, y, z our initial inequality reads

$$\frac{1}{3}\left\{\frac{2y+2z-x}{x}+\frac{2x+2z-y}{y}+\frac{2x+2y-z}{z}\right\}\geq 3,$$

or even

$$2\left(\frac{x}{y}+\frac{y}{x}\right)+2\left(\frac{y}{z}+\frac{z}{y}\right)+2\left(\frac{x}{z}+\frac{z}{x}\right)\geq 12. \qquad (5)$$

By AM-GM inequality we have

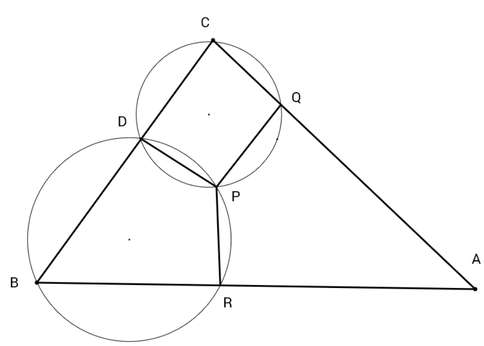
$$\frac{x}{y}+\frac{y}{x}\geq 2, \quad \frac{y}{z}+\frac{z}{y}\geq 2, \quad \frac{x}{z}+\frac{z}{x}\geq 2.$$

Adding the above inequalities we find (5) which proves our initial inequality.

5. ABC is an acute triangle and D is a point on the segment BC. Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ passing through B, D and C, D respectively intersect for the second time at P, where P lies inside of triangle ABC. Denote by R the intersection of $\mathcal{C}_1$ and AB and by Q the intersection of $\mathcal{C}_2$ and AC.

Prove that P lies on the circumcircle of triangle QAR.

Solution. The quadrlateral $DPRB$ is cyclic, so $\angle DPR = 180^o - \angle DBR$.



Since $CDPQ$ is cyclic, $\angle DPQ = 180 -^o - \angle DCQ$. Now,

$$\begin{aligned}\angle QPR &= 360^o - (\angle DPR + \angle DPQ)\\ &= \angle DBR + \angle DCQ\\ &= 180^o - \angle CAB\\ &= 180^o - \angle QAR,\end{aligned}$$

so $QPRA$ is also cyclic.

6. 25 boys and 25 girls are at a party. Each boy likes at least 13 girls, and each girl likes at least 13 boys.

Show that there must be a boy and girl at the party who like each other.

Solution. Consider all pairs (r, s) where r is a boy, s is a girl, and r likes s. For any boy r, there are at least 13 such pairs. Therefore there are at least $25 \cdot 13$ such pairs in total. From this we conclude that, since there are 25 girls, there must be a girl who is liked by at least 13 boys. If this particular girl does not like any of these boys, then she can only like some subset of the remaining 12 boys. But from the statement of the problem we know that she likes at least 13 boys, so this is a contradiction. We conclude that there must be a boy and a girl who like each other.

7. On sides AB, BC and CA of triangle ABC we consider the points M, N and P respectively such that

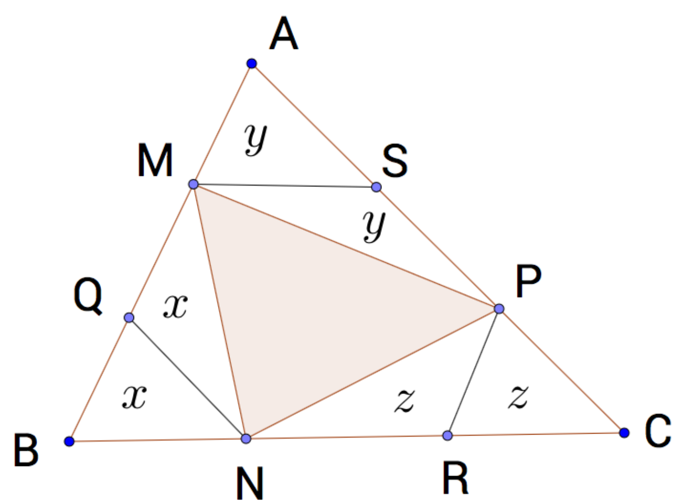
$$\frac{AM}{MB} = \frac{BN}{NC} = \frac{CP}{PA} = \frac{1}{2}.$$

Prove that:

$$(a)\ [AMN] = \frac{1}{9}[ABC] \qquad\qquad (b)\ [MNP] = \frac{1}{3}[ABC].$$

Solution. (a) Denote by Q, R, S the midpoints of BM, CN and AP respectively.

Using the property of the median in a triangle we have $[BNQ] = [MNQ] = x$. Since triangles BNQ and BCA are similar, it follows that $x = \frac{1}{9}[ABC]$. Now, NM is median in triangle NQA so $[AMN] = [MNQ] = x = \frac{1}{9}[ABC]$.



(b) As before $y = z = \frac{1}{9}[ABC]$ so $[MNP] = [ABC] - 2x - 2y - 2z = \frac{1}{3}[ABC]$.

8. Richard and nine other people are standing in a circle. All ten of them think of an integer (that may be negative) and whisper their number to both of their neighbours. Afterwards, they each state the average of the two numbers that were whispered in their ear.

Richard states the number 10, his right neighbour states the number 9, the next person along the circle states the number 8, and so on, finishing with Richard's left neighbour who states the number 1.

What number did Richard have in mind?

Solution. Denote the number that Richard came up with by c_{10}, the number of his right neighbour by c_9, the number of the next person along the circle by c_8, and so on, finishing with the number of Richard's left neighbour which we denote by c_1.

From the data we deduce that

$$c_{10} + c_8 = 2 \cdot 9 = 18,$$

$$c_8 + c_6 = 2 \cdot 7 = 14,$$

$$c_6 + c_4 = 2 \cdot 5 = 10,$$

$$c_4 + c_2 = 2 \cdot 3 = 6,$$
$$c_2 + c_{10} = 2 \cdot 1 = 2.$$

Adding up these equations yields $2(c_2 + c_4 + c_6 + c_8 + c_{10}) = 50$, hence $c_2 + c_4 + c_6 + c_8 + c_{10} = 25$.

Finally, we find $c_{10} = (c_2 + c_4 + c_6 + c_8 + c_{10}) - (c_2 + c_4) - (c_6 + c_8) = 25 - 6 - 14 = 5$.

Therefore, Richard originally thought of the number 5.

9. For each positive integer n let $s_n = n! + 20!$.

   (a) Let $q > 20$ be a prime number. Prove that there are only a finite number of positive integers k for which q divides s_k.

   (b) Find with proof all prime numbers p for which there exists a positive integer m such that p divides s_m and s_{m+1}.

Solution. (a) Suppose $q > 20$ is a prime number and that q divides s_k. Note that q does not divide $20!$ so k does not divide $k!$. Hence $q > k$. but then, if $\ell \geq q$ is an integer, then q divides $\ell!$, so, since q does not divide $20!$, we have that q does not divide s_ℓ. Thus, the set of positive integers k for which q divides s_k is contained in the $\{21, 22, \ldots, q-1\}$, so, it is finite.

(b) We claim that $p \leq 19$.

Note first that $s_{m+1} - s_m = (m+1)! - m! = m \cdot m! = m(m! + 20) - m \cdot 20!$. Thus, since p divides s_m and s_{m+1}, it must divide their difference, so in the end, p divides $m \cdot 20!$. If p divides m then p divides $m!$ and since p divides s_m it must divide $20!$. Hence p divides $20!$ and since p is prime it follows $p \leq 19$.

Suppose $p \leq 19$. Then p divides $s_{19} = 19! + 20!$ and p divides $s_{20} = 20! + 20!$. Thus, all the required numbers are $2, 3, 5, 7, 11, 13, 17, 19$.

10. For a real number x denote by $[x]$ the greatest integer not exceeding x.

    (a) Find with proof all positive integers k for which $[\sqrt[3]{k^3 + 20k}] \neq k$.

    (b) Prove that if n is a positive integer, then $\left[n + \sqrt{n} + \frac{1}{2}\right]$ is not the square of an integer.

Solution. (a) Note that $k^3 + 20k < (k+1)^3 = k^3 + 3k^2 + 3k + 1$ if and only if $3k^2 - 17k + 1 > 0$.

The quadratic equation $3x^3 - 17x + 1 = 0$ has two real roots

$$\alpha = \frac{17 - \sqrt{17^2 - 12}}{6} \quad \text{and} \quad \beta = \frac{17 + \sqrt{17^2 - 12}}{6}$$

and

$$3k^2 - 17k + 1 = 3(k - \alpha)(k - \beta) > 0 \quad \text{if and only if} \quad k < \alpha \quad \text{or} \quad k > \beta.$$

Note that

$$0 < \alpha < 1 \quad \text{and} \quad 5 < \beta < 6.$$

Hence, for $k \geq 6$ we have

$$k < \sqrt[3]{k^3 + 20k} < k + 1$$

which yields $[\sqrt[3]{k^3 + 20k}] = k$. On the other hand, for $k = 0, 1, 2, 3, 4, 5$ we check separately that $[\sqrt[3]{k^3 + 20k}] \neq k$.

(b) Let $m = [\sqrt{n}]$, so $m^2 \leq n \leq (m+1)^2$. Let $r = n - m^2$ so $n = m^2 + r$ and $0 \leq r \leq 2m$

Observe that

$$\left(m + \frac{r}{2m}\right)^2 = m^2 + r + \frac{r^2}{4m^2} > m^2 + r = n$$

so $\sqrt{n} < m + \frac{r}{2m}$.

If $0 \leq r \leq m$ then $\frac{r}{2m} \leq \frac{1}{2}$ and

$$m^2 + m < n + \sqrt{n} + \frac{1}{2} < m^2 + r + m + 1 \leq m^2 + 2m + 1$$

which shows that

$$m^2 < \left[n + \sqrt{n} + \frac{1}{2}\right] < (m+1)^2,$$

and $\left[n+\sqrt{n}+\frac{1}{2}\right]$ is not the square of an integer in this case.

Finally, if $m+1 \le r \le 2m$ then

$$\sqrt{n} \ge \sqrt{m^2+m+1} > m + \frac{1}{2}$$

and then

$$n+\sqrt{n}+1 > (m^2+m+1)m + \frac{1}{2} + \frac{1}{2} = (m+1)^2+1.$$

By direct calculations we have

$$n+\sqrt{n}+\frac{1}{2} < m^2+2m+m+1+\frac{1}{2} < (m+2)^2.$$

Hence

$$(m+1)^2+1 < n+\sqrt{n}+\frac{1}{2} < (m+2)^2$$

which yields

$$(m+1)^2+1 \le \left[n+\sqrt{n}+\frac{1}{2}\right] < (m+2)^2$$

and again $\left[n+\sqrt{n}+\frac{1}{2}\right]$ is not the square of an integer.